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DEGREE AND LOCAL CONNECTIVITY IN DIGRAPHS

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It is shown that there is a digraph D of minimum outdegree $12m$ and $\max_{x \neq y} \mu(x, y; D) = 11m$, but every digraph D of minimum outdegree n contains vertices $x \neq y$ with $\lambda(x, y; D) \geqq n-1$, where $\mu(x, y; D)$ and $\lambda(x, y; D)$ denote the maximum number of openly disjoint and edge-disjoint paths, respectively.

In [7] it was shown that every finite (non-trivial) graph G contains vertices $x \neq y$ such that $\mu(x, y; G) = \min \{e(x; G), e(y; G)\}$, where $e(x; G)$ denotes the degree of the vertex x in G and $\mu(x, y; G)$ is the maximum number of openly disjoint paths joining x and y in G. Especially, if every vertex has degree at least n, then there are vertices x and y connected by n openly disjoint paths. In the present paper we study an analogous question for directed graphs: If every vertex x of a finite directed graph or multigraph D has outdegree $e^+(x; D) = n$, what can one say about $\max_{x \neq y} \mu(x, y; D)$ and $\max_{x \neq y} \lambda(x, y; D)$, where $\mu(x, y; D)$ and $\lambda(x, y; D)$ are the maximum number of openly disjoint (continuously directed) paths and edge-disjoint paths from x to y in D, respectively. As expected, these problems are more difficult than in the undirected case. For $\max \mu(x, y; D)$, we can only show that for every positive number m there is a finite digraph D with $e^+(x; D) \geqq 12m$ for every vertex x, but $\max_{x \neq y} \mu(x, y; D) = 11m$. I do not even know if $\max \mu(x, y; D)$ increases with $\min e^+(x; D)$. For finite directed multigraphs D, always $\max_{x \neq y} \lambda(x, y; D) > \min e^+(x; D)/2$, whereas for every positive integer n, there are finite directed multigraphs D such that $\min_x e^+(x; D) = n$ and $\max \lambda(x, y; D) = \lfloor 2n/3 \rfloor + 1$. If D does not have parallel edges, then we can prove $\max \lambda(x, y; D) \geqq \min e^+(x; D) - 1$, but I could not settle if even $\max \lambda(x, y; D) \geqq \min e^+(x; D)$ is valid.

First let us state some concepts and notations. All graphs and multigraphs considered here are supposed to be finite without loops. A directed multigraph may have parallel edges of the same direction, but a digraph does not. The vertex set and the edge set of D are denoted by $V(D)$ and $E(D)$, respectively; furthermore, $|D| := |V(D)|$. The set of edges from x to y is denoted by (x, y) and if there is only

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one, we identify (x, y) with this edge; in the undirected case we write $[x, y]$ for an edge. For $k\in(x, y)$, $V(k):=\{x, y\}$. For $A, B\subseteq V(D)$, let $D(A)$ be the subgraph spanned by A, $E(A, B; D):=\bigcup\{(a, b): a\in A, b\in B\}$, $E^+(A; D):=E(A, V(D)-A; D)$, $E^-(A; D):=E^+(V(D)-A; D)$, and $e^*(\dots):=|E^*(\dots)|$ for $*=\ , +, -$. (In these notations we write a instead of $\{a\}$.) For $a\in V(D)$, $N^+(a; D):=\{x\in V(D-a): (a, x)\neq\emptyset\}$ and for $A\subseteq V(G)$ in a graph G, $N(A; G):=\{x\in V(G)-A: \exists_{a\in A}[a, x]\in \in E(G)\}$. We call a directed multigraph D outregular of degree n, if $e^+(x; D)=n$ for all $x\in V(D)$ and we define $\delta_n^+(A; D):=\sum_{a\in A}\max(0, n-e^+(a; D))$ for $A\subseteq V(D)$ and $\delta_n^+(D):=\delta_n^+(V(D); D)$; "inregular" and $\delta_n^-(D)$ are defined analogously. In a directed multigraph, the terms "path" and "circuit" always mean continuously directed path and circuit, and an x, y-path is a path from x to y. A digraph consisting of n vertices and $n(n-1)$ edges is denoted by $\vec{K}_n$.

Let us first construct a digraph D which is outregular of degree n, but has $\max\mu(x, y; D)<n$. For $n=12m$ and any positive integer $d\leqq m$ let D_0 be a digraph with $|D_0|=n$ which is outregular and inregular of degree $n-d$ and contains $6m$ edges $k_1=(x_1, y_1), \dots, k_{n/2}=(x_{n/2}, y_{n/2})$ with $\{x_i, y_i\}\cap\{x_j, y_j\}=\emptyset$ for $i\neq j$. (For instance, delete from $\vec{K}_n$ $(d-1)$ edge-disjoint hamiltonian circuits such that the remaining digraph has a "1-factor".) Delete the edges $k_1, \dots, k_{6m}$ and add a new vertex z_0 and the edges (x_i, z_0) and (z_0, y_i) for $i=1, \dots, 6m$. Add $3d$ further vertices $z_1, \dots, z_{3d}$ and edges from $V(D_0)$ to $z_1, \dots, z_{3d}$ in such a way that we get a digraph $\bar{D}_0$ with $e^-(z_i; \bar{D}_0)=4m$ for $i=1, \dots, 3d$ and $e^+(x; \bar{D}_0)=n$ for all $x\in V(D_0)$. If we perform this procedure with the same vertices $z_0, z_1, \dots, z_{3d}$ and $2(3d+1)$ disjoint copies $D_0, D_1, \dots, D_{6d+1}$ of D_0 $6d+2$ times altogether in such a way that every vertex z_i plays the same role as z_0 above exactly twice, we get a digraph D outregular of degree n, where $e^-(x; D)=n-d$ for all $x\in \bigcup_{i=0}^{6d+1} V(D_i)$ and $e^-(z_i; D)>n$ for $i=0, 1, \dots, 3d$. It is easy to see that D has the desired property $\max_{x\neq y}\mu(x, y; D)=n-d<n=\min_x e^+(x; D)$.

In a similar manner it is possible to give an example of a digraph D with $\delta_n^+(D)=\delta_n^-(D)=0$, but $\max\mu(x, y; D)<n$. To this, we consider the digraph D constructed above for $n=12m^2$ and $d=m$. May $\bar{D}$ arise from D by adding $(6m+2)m$ further vertices $a_i(i=0, 1, \dots, 6m^2+2m-1)$ and $10m^2$ edges from a_i to $D_{\lfloor i/m\rfloor}$ and $2m^2$ edges from a_i to $D_{\lfloor i/m\rfloor+1}$ (the indices of the Ds modulo $6m+2$) in such a way that $e^-(x; \bar{D})=n$ for all $x\in\bigcup_{i=0}^{6m+1} V(D_i)$. Taking a (disjoint) dual digraph $\bar{D}'$ of $\bar{D}$, where the vertex a_i' may correspond to a_i, and identifying a_i with a_i' for $i=0, \dots, 6m^2+2m-1$, we get a digraph H with $\delta_n^+(H)=\delta_n^-(H)=0$, but $\max\mu(x, y; H)=n-m$.

It was proved in [5] and [6] that for given n, there is an m such that every (undirected, finite) graph of minimum degree m contains an n-connected subgraph and a subdivision of the complete graph K_n. The corresponding statements for digraphs do not hold, as the example below shows. It was conjectured by Y. O. Hamidoune in [3], that every digraph D with $\delta_n^+(D)=\delta_n^-(D)=0$ contains an edge (x, y) such that $\mu(y, x; D)\geqq n$. This conjecture was disproved by C. Thomassen in [9]. Modifying his counterexample, we shall show now that for every n there is a finite digraph D with $\delta_n^+(D)=\delta_n^-(D)=0$ and with the property that $\mu(x, y; D)\leqq$

$\leqq 1$ or $\mu(y, x;\ D) \leqq 1$ for all vertices $x \neq y$ in D. (But it is easily shown that every directed multigraph D with $\delta_2^+(D) \leqq 1$ contains vertices $x \neq y$ which are connected by 3 openly disjoint paths, two x, y-path and one y, x-path.)

Let n be given and let D be a finite digraph satisfying the following conditions:

(1) $(x, y) \in E(D) \to (y, x) \notin E(D)$ (and, at pleasure, in addition $\mu(y, x;\ D) \leqq 2$);
(2) $(x, y) \in E(D) \wedge \mu(y, x;\ D) \geqq 2 \to e^+(x;\ D) \geqq n$;
(3) If C_1 and C_2 are circuits in D with $V(C_1) \cap V(C_2) = \{x\}$, then $e^+(x;\ D) \geqq n$;
(4) $\mu(x, y;\ D) \geqq 2 \to \mu(y, x;\ D) \leqq 1$.

If $m := \min\{e^+(x;\ D)\colon x \in V(D)\} < n$, we shall construct a digraph D' with $\min\{e^+(x;\ D')\colon x \in V(D')\} \geqq \min\{e^+(x;\ D)\colon x \in V(D)\}$ which meets also the conditions (1) to (4) and the number of vertices with outdegree m in D' is smaller than the number of such vertices in D. Let us suppose $m < n$ and let x_0 be a vertex of D with $e^+(x_0;\ D) = m$. The digraph L may consist of a path $y_0, y_1, \ldots, y_n$ of length n added the $n-1$ edges $(y_0, y_2), \ldots, (y_0, y_n)$ and $n-m$ circuits $C_1, \ldots, C_{n-m}$ of length $\geqq 3$ such that $V(C_i) \cap V(C_j) = \{y_n\}$ for all $i \neq j$ and $V(C_i) \cap \{y_0, \ldots, y_{n-1}\} = \emptyset$ for all i. We assume $V(L) \cap V(D) = \emptyset$. The digraph D' may arise from $D \cup L$ by adding the edge (x_0, y_0) and all the edges from $V(L - y_0)$ to $N^+(x_0;\ D)$. There is no difficulty to check that also D' has the four properties (1) to (4). As $e^+(x;\ D') > m$ for all $x \in V(L)$, by successive application of this construction, in a finite number of steps we can get a digraph $\bar{D}$ with $\delta_n^+(\bar{D}) = 0$ and $\mu(x, y;\ \bar{D}) \leqq 1$ or $\mu(y, x;\ \bar{D}) \leqq 1$ for all vertices $x \neq y$. Taking a (disjoint) dual $\bar{D}'$ of $\bar{D}$ and adding all the edges from $\bar{D}'$ to $\bar{D}$ to the digraph $\bar{D}' \cup \bar{D}$, we get a digraph with the asserted properties. (It is also obvious, that a digraph D with $\delta_n^+(D) = \delta_n^-(D) = 0$ does not necessarily contain a 2-edge-connected subgraph.)

The following problem remains: Given any positive integer n, is there an m such that every finite digraph D outregular of degree m contains a subdivision of the acyclic tournament of order n. Of course, this would imply that existence of vertices $x \neq y$ with $\mu(x, y;\ D) \geqq n-1$.

Whereas we have found only negative results for the connection of the function μ with the minimum outdegree of a digraph, we can give some positive ones for λ.

Theorem 1. *Let D be a finite directed multigraph with $\delta_n^+(D) < n$ for a positive integer n. Then there are vertices $x \neq y$ and $n+1$ edge-disjoint paths $P_1, \ldots, P_{n+1}$ in D such that P_i is an x, y-path or a y, x-path for $i = 1, \ldots, n+1$.*

Proof. We use induction on the number of vertices. $\delta_n(D) < n$ implies $|D| \geqq 2$. We may assume $e^+(x;\ D) \leqq n$ for all $x \in V(D)$. There is a vertex z with $e^-(z;\ D) \leqq e^+(z;\ D)$; set $d := n - e^+(z;\ D) \geqq 0$. We define a bipartite graph G by $V(G) := E^-(z;\ D) \cup E^+(z;\ D)$ and $k, k' \in V(G)$ adjacent iff $\{k, k'\} \cap E^-(z;\ D) \neq \emptyset$, $\{k, k'\} \cap E^+(z;\ D) \neq \emptyset$, and $V(k) \neq V(k')$. Let us first suppose that there is a matching M in G with $|M| \geqq e^-(z;\ D) - d$, say, $M = \{[k_i, k_i']\colon i = 1, \ldots, m\}$ with $k_i \in (x_i, z)$ and $k_i' \in (z, y_i)$. Let D' arise from $D - z$ by adding a new edge $\bar{k}_i$ from x_i to y_i for every $i = 1, \ldots, m$. Then D' has no loops and $\delta_n^+(D') \leqq \delta_n^+(D) < n$. By induction, there are vertices $x \neq y$ and $n+1$ edge-disjoint paths joining x and y in D'. Substituting k_i and k_i' for $\bar{k}_i$, these paths also deliver $n+1$ edge-disjoint paths joining x and y in D. Therefore, we may assume that there is no matching M in G with $|M| \geqq e^-(z;\ D) - d$. Then, by Hall's theorem (cf. [8]), there is a subset

$A \subseteqq E^-(z;\ D)$ with $|A| > |N(A;\ G)| + d$. If $A \subsetneqq (x, z)$ for all $x \in V(D-z)$, then $N(A;\ G) = E^+(z;\ D)$, which implies the contradiction $e^-(z;\ D) \geqq |A| > e^+(z;\ D)$. Hence, there is an $x_0 \in V(D-z)$ such that $A \subseteqq (x_0, z)$. But then there are $n+1$ edge-disjoint paths connecting x_0 and z in D, because $|(x_0, z)| + |(z, x_0)| > |N(A;\ G)| + d + |(z, x_0)| = e^+(z;\ D) + d = n$. ∎

Corollary. *Every finite directed multigraph D with $\delta_n^+(D) < n$ contains vertices $x \neq y$ such that $\lambda(x, y;\ D) \geqq \lfloor n/2 \rfloor + 1$.* ∎

Simple examples show that this Corollary is best possible. But in a directed multigraph D with $\delta_n^+(D) = 0$, there are vertices $x \neq y$ such that $\lambda(x, y;\ D) \geqq \lfloor n/2 \rfloor + 2$, for $n = 3$ and all $n \geqq 5$ (whereas Theorem 1 remains sharp also for $\delta_n^+(D) = 0$). On the other hand, for every n, there are directed multigraphs D such that $\delta_n^+(D) = 0$ and $\max\{\lambda(x, y;\ D)\colon x \neq y\} = \lfloor 2n/3 \rfloor + 1$. To see this, consider the directed multigraph D' displaced in figure 1. If we stick together n copies of D' at z, we get a directed multigraph D as wanted. By the way, in a directed multigraph D with $e^+(x;\ D) = e^-(x;\ D)$ for all $x \in V(D)$, there are $x \neq y$ such that $\lambda(x, y;\ D) = \min\{e^+(x;\ D), e^-(y;\ D)\}$. This follows, for instance, from the generalization of the algorithm of Gomory and Hu for the flow function of a graph, given by R. P. Gupta in [1].

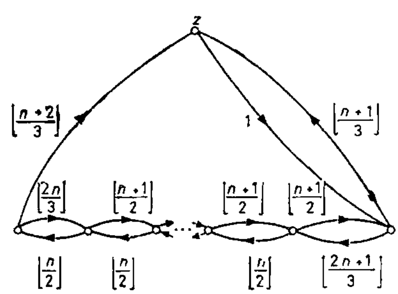

Fig. 1

If we forbid multiple edges, we can prove a result more precise.

Theorem 2. *Let D be a finite digraph with $\delta_n^+(D) < 2(n-1)$ and $|D| > 1$. Then there are vertices $x \neq y$ in D such that $\lambda(x, y;\ D) \geqq n-1$.*

Proof. $\delta_n^+(D) < 2(n-1)$ and $|D| > 1$ implies $n \geqq 2$ and $|D| \geqq n$, as easily checked. Hence, there is a vertex a in D with $e^+(a;\ D) \geqq n-1$. We may assume $e^+(a;\ D) \leqq n$ and $\lambda(a, x;\ D) \leqq n-2$ for all $x \in V(D-a)$. By the edge-form of Menger's theorem (cf. Chap. IV, § 2 in [2]), there is a $U(x) \subseteqq V(D-a)$ such that $x \in U(x)$ and $\lambda(a, x;\ D) = e^-(U(x);\ D)$. From the sets $U(x)$, we choose a minimal cover $U_1, \dots, U_k$ of $V(D-a)$. For $i = 1, \dots, k$, be $V_i := U_i - \bigcup\{U_j\colon j \neq i\}$. Since $U_1, \dots, U_k$ is a minimal cover, $V_i \neq \emptyset$ for all i. We have the following inequalities

$$(1) \qquad k(n-2) \geqq \sum_{i=1}^{k} e^-(U_i;\ D) = \sum_{i=1}^{k} e^-(U_i;\ D-a) + \sum_{i=1}^{k} e(a, U_i;\ D)$$

$$\geqq \sum_{i=1}^{k} e^-(U_i;\ D-a) + 2e^+(a;\ D) - \sum_{i=1}^{k} e(a, V_i;\ D),$$

because the edges $E^+(a;\, D)-\bigcup_{i=1}^{k} E(a, V_i;\, D)$ are counted twice, at least, in $\sum_{i=1}^{k} e(a, U_i;\, D)$. On the other hand, we have

$$\sum_{i=1}^{k} e^+(V_i;\, D-a) \leqq \sum_{i=1}^{k} e^-(U_i;\, D-a), \tag{2}$$

because $E^+(V_i;\, D-a) \subseteqq \bigcup_{j\neq i} E^-(U_j;\, D-a)$ and the sets V_i, hence the sets $E^+(V_i;\, D-a)$, are disjoint. From these inequalities (1) and (2), we get

$$\sum_{i=1}^{k} \left(e^+(V_i;\, D-a)-e(a, V_i;\, D)+\delta_n^+(V_i;\, D)\right) < k(n-2)-2e^+(a;\, D)+2(n-1)$$
$$\leqq k(n-2).$$

Hence, there is an i_0 such that $e^+(V_{i_0};\, D-a)-e(a, V_{i_0};\, D)+\delta_n^+(V_{i_0};\, D)<n-2$. For $D_0:=D(V_{i_0}\cup\{a\})$, that means $\delta_n^+(D_0)=\delta_n^+(V_{i_0};\, D)+e^+(V_{i_0};\, D-a)+n-$ $-e(a, V_{i_0};\, D)<2(n-1)$. Hence D_0 satisfies the conditions of the theorem. As $k\geqq 2$, we have $|D_0|<|D|$ and, therefore, we can finish the proof by induction.

Remark. I should conjecture that a digraph D (perhaps with a few multiple edges) with $\delta_n^+(D)<n$ contains vertices $x\neq y$ such that $\lambda(x, y;\, D)\geqq n$. Considering the decomposition used by L. Lovász in [4], it is possible to reduce such a digraph with $\max \lambda(x, y;\, D)<n$ by contraction, but only by generating multiple edges. But for directed multigraphs such a theorem does not hold, as we have seen above. This ist he dilemma which I could not manage.

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